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Using G-algebras for Schur index computation

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Abstract

Conditions for the existence of a bijective character correspondence that preserves Schur indices are given, based on the theory of central simple G-algebras over fields of characteristic 0. It is then shown how this applies to Schur index computation in the case of finite solvable groups.

Keywords: G-algebras; Schur index; Characters

1. Introduction

Let G be a finite group, χ an irreducible character of G, and K an algebraic number field. The standard theoretical method for computing the Schur index $m_K(\chi)$ associated with χ over K utilizes the Brauer–Witt theorem to reduce to a situation that can be handled using Clifford theory and number theory, the latter being used to solve norm equations in every completion of K. Although it is tedious to solve these norm equations, it is the Brauer–Witt theorem which is the main barrier to obtaining an effective theory for Schur index computation, since this theorem only indicates the existence of subgroups of G that produce the prime power divisors of $m_K(\chi)$. In this paper we formulate a new constructive reduction (or series of reductions) that can replace the Brauer–Witt theorem in some cases.

The present paper was motivated by the theory of G-algebras as developed by Turull [6], and the author's impression that it should be possible to base a Schur index reduction on this theory. Turull gave conditions for a bijective character correspondence to exist between certain sets of irreducible characters of groups G and H that will preserve the

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Schur indices of corresponding irreducible characters. In Section 2 we review Turull's theory. In Section 3, we show that for any normal subgroup N of a finite group G, there is a unique G/N-algebra equivalence class determined by all characters which are quasiprimitive with respect to a field K and which lie over a common irreducible character of N. An investigation of the structure of the G/N-algebras determining this class leads us to conclude that Turull's–Schur index preserving bijection can be established whenever there are characters $\chi \in \mathrm{Irr}(G)$ and $\xi \in \mathrm{Irr}(H)$ that have minimal possible degree, are quasiprimitive with respect to K, and generate the same element of the Brauer group of K. In Section 4, we describe conditions on characters of solvable groups that ensure this situation occurs, and obtain a reduction which extends the main result of [1]. The results of Section 4 require Philip Hall's structure theorem for solvable groups having all normal abelian subgroups cyclic, the Isaacs–Dade character correspondence as presented in [2], and stable Clifford theory.

2. Background

Turull's "Clifford theory with Schur indices" is based on the notion of equivalence classes of central simple G-algebras developed in [6], where all of the following definitions and results appear. The reader is referred to this paper for proofs of the results in this section.

Let K be a field of characteristic 0, G be a group, and let A be a finite-dimensional associative K-algebra. We say that A is a *G-algebra* over K if there is an action of G by K-algebra automorphisms on A. The center $Z(A)$ of the G-algebra A is naturally G-invariant, and we say that A is a *central* G-algebra over K if $C_{Z(A)}(G) = K$. A is a *simple* G-algebra if the only two-sided G-invariant ideals of A are 0 and A. Two G-algebras over K are *G-algebra isomorphic* if there is a K-algebra automorphism from one to the other that commutes with the respective G-actions.

An important elementary example of a central simple G-algebra is a *trivial* G-algebra, which is defined to be any G-algebra which is G-algebra isomorphic to $\mathrm{End}_K(V)$, where V is a finite-dimensional KG-module. If $\rho : KG \to \mathrm{End}_K V$ is a representation affording V, then the G-action on $\mathrm{End}_K V$ is given by ${}^g f = \rho(g) f \rho(g^{-1})$, for all $g \in G$, for all $f \in \mathrm{End}_K V$. The trivial G-algebras are an important class of central simple G-algebras because they are also central simple algebras. When one forms the tensor product of two central simple G-algebras over K, one does not necessarily obtain a central simple G-algebra. However, if one of the central simple G-algebras is also a central simple K-algebra (such as a trivial G-algebra), then their tensor product over K will be a central simple G-algebra [6, Lemma 1.4]. We say that two central simple G-algebras A and B over a field K are *equivalent* if there are trivial G-algebras E_1 and E_2 such that $A \otimes_K E_1$ and $B \otimes_K E_2$ are G-algebra isomorphic. This defines an equivalence relation on the class of central simple G-algebras over K, and the collection of equivalence classes is denoted by $S(G, K)$.

The following results are crucial for Turull's approach.

Lemma 2.1 [6, Lemma 2.1]. *Suppose K has characteristic 0, and let V be an irreducible KG-module. Suppose that $\chi \in \mathrm{Irr}(G)$ is an irreducible constituent of the character afforded by V. Let N be any normal subgroup of G.*

Then $\mathrm{End}_{KN} V$ is a central simple G/N-algebra over $K(\chi_N)$ in a natural way.

The G/N-action on $\mathrm{End}_{KN} V$ is defined using the representation $\rho : KG \to \mathrm{End}_K V$ affording V. For all $f \in \mathrm{End}_{KN} V$, for all $\bar{g} \in G/N$, let ${}^{\bar{g}}f = \rho(g) f \rho(g^{-1})$, for any coset representative $g \in G$ of $\bar{g}$. Required for Turull's proof of Lemma 2.1 is the following observation, which will be required in Section 3.

Proposition 2.1. *$Z(\mathrm{End}_{KN} V)$ is isomorphic to the image of $Z(KN)$ under the representation $\rho : KG \to \mathrm{End}_K V$.*

If U is any KG-module whose character ψ has the property that ψ_N is a rational multiple of χ_N, then we say that U is *χ-quasihomogeneous with respect to N*.

Lemma 2.2 [6, Proposition 2.3]. *If U is a χ-quasihomogeneous KG-module with respect to N, and $F = K(\chi_N)$, then $\mathrm{End}_{FN} U$ is a central simple G/N-algebra. Furthermore, if U_1 and U_2 are both χ-quasihomogeneous with respect to N, then $\mathrm{End}_{FN} U_1$ and $\mathrm{End}_{FN} U_2$ are equivalent G/N-algebras.*

This allows one to define, for each $\chi \in \mathrm{Irr}(G)$, an element $[\![\chi]\!]$ in $S(G/N, K(\chi_N))$ by setting $[\![\chi]\!]$ to be the equivalence class of $\mathrm{End}_{FN} V$, for any KG-module V which is χ-quasihomogeneous with respect to N. The strength of Turull's approach is the control of Schur indices that occurs when one has equality of two G/N-algebra equivalence classes corresponding to irreducible characters.

Theorem 2.1 [6, Theorem 3.5]. *Suppose that G and H are finite groups that have isomorphic sections $G/N \cong H/M$. For each subgroup S of G containing N, let $\tau(S)$ be the subgroup of H containing M that corresponds to S under a fixed isomorphism $G/N \cong H/M$. Suppose that there are irreducible characters $\chi \in \mathrm{Irr}(G)$ and $\xi \in \mathrm{Irr}(H)$ that satisfy $K(\chi_N) = K(\xi_M)$ and $[\![\chi]\!] = [\![\xi]\!]$ in $S(G/N, K(\chi_N))$.*

Then there is a bijection between $\mathrm{Irr}(S|\chi_N)$ and $\mathrm{Irr}(\tau(S)|\xi_M)$ that preserves elements of the corresponding Brauer groups, and hence Schur indices.

3. Equivalence classes of G-algebras for quasiprimitive characters

In this section we consider the G-algebras defined in Turull's approach that arise from characters that are quasiprimitive with respect to a given field K of characteristic 0. We say that $\chi \in \mathrm{Irr}(G)$ is quasiprimitive with respect to K if for all normal subgroups N of G, χ_N is an integer multiple of a K-irreducible character of N—the sum of all $\mathrm{Gal}(K(\chi,\varphi)/K(\chi))$-conjugates of a single $\varphi \in \mathrm{Irr}(N|\chi)$. In the situation where $K(\chi)=K$, the group algebra KG contains the complex idempotent

$$e_\chi = \frac{1}{|G|}\sum_{g\in G}\chi(1)\chi\left(g^{-1}\right)g,$$

and χ is quasiprimitive with respect to K if and only if the subalgebra KNe_χ is a simple K-algebra whenever N is a normal subgroup of G.

Proposition 3.1. *Let V be an irreducible KG-module. Let $\chi \in \mathrm{Irr}(G)$ be a complex constituent of the character afforded by V, and suppose that $K(\chi) \subseteq K$. Let N be a normal subgroup of G for which V_N is a homogeneous KN-module. Then*

(i) $\mathrm{End}_{KN}\, V$ *is a simple K-algebra*; *and*
(ii) $\mathrm{End}_{KG}\, V$ *is a central simple G/N-algebra over K.*

Proof. The irreducible KG-module V affords $m\chi$, where $m=m_K(\chi)$. Since V_N is a homogeneous KN-module, χ_N is a multiple of the Galois conjugacy class sum of some $\varphi\in\mathrm{Irr}(N|\chi)$. In this situation, KNe_χ is a simple subalgebra of the simple algebra KGe_χ, and $Z(KNe_\chi)\cong K(\varphi)$. By Proposition 2.1, $Z(\mathrm{End}_{KN}\, V)$ is isomorphic to the the center of $\rho(KN)$, where $\rho : KG \to \mathrm{End}_K(V)$ is a representation affording V. Since $\rho(KN)$ is naturally isomorphic to the opposite ring of KNe_χ, it follows that $Z(\mathrm{End}_{KN}\, V)$ is a field, which proves (i).

The restriction of the natural action of G/N on $\mathrm{End}_{KN}\, V$ to $\mathrm{End}_{KG}\, V$ gives a trivial action of G/N on $\mathrm{End}_{KG}\, V$. Since $\mathrm{End}_{KG}\, V$ is a central simple K-algebra upon which G/N acts trivially, (ii) follows. □

A simple generalization of [8, Lemma 2.1] is convenient for computing dimensions.

Lemma 3.1. *Let N be a normal subgroup of G. Suppose that V is a KG-module affording a character η of G for which $K(\eta_N)=K$. Then* $\dim_K(\mathrm{End}_{KN}\, V)=(\eta_N,\eta_N)$.

Proof. Since $\mathrm{End}_K V \cong (V\otimes_K V^*)$ as KG-modules, it follows that $\mathrm{End}_{KN} V \cong C_{(V\otimes V^*)}(N)$. The dimension of $C_{(V\otimes V^*)}(N)$ over K is the multiplicity of the trivial character 1_N in $V\otimes V^*$ when it is considered as a KN-module. This is $(1_N,(\eta\bar\eta)_N) = (1_N,\eta_N\overline{\eta_N})=(\eta_N,\eta_N)$, as claimed. □

The next theorem describes $\mathrm{End}_{KN}\, V$ for K-quasiprimitive modules.

Theorem 3.1. *Let N be a normal subgroup of G, $\varphi\in\mathrm{Irr}(N)$, and $I=I_G(\varphi)$. Suppose that $\chi\in\mathrm{Irr}(G)$ is quasiprimitive with respect to the field K. Let α be the unique element of $\mathrm{Irr}(I|\varphi)$ such that $\alpha^G=\chi$. Let V be an irreducible KG-module affording $m\chi$, with $m=m_K(\chi)$. Then*

(i) $\dim_K(\mathrm{End}_{KN}\, V)=m^2d^2n$, *where $d=(\alpha(1))/(\varphi(1))$ and $n=[G:I]$*; *and*

(ii) $\mathrm{End}_{KN} V \cong \mathrm{End}_{KG} V \otimes_K C_{\mathrm{End}_{KN} V}(\mathrm{End}_{KG} V)$ *as* G/N*-algebras. The algebra* $C_{\mathrm{End}_{KN} V}(\mathrm{End}_{KG} V) = A$ *is a central simple* G/N*-algebra with the properties* $Z(A) \cong K(\varphi)$, $C_A(G/N) \cong K$, *and* $\dim_K(A) = d^2 n$.

Proof. Let T be a transversal of I in G. Since V affords $m\alpha^G$, we have by Lemma 3.1 that

$$\dim_K(\mathrm{End}_{KN} V) = \left((m\alpha^G)_N, (m\alpha^G)_N\right) = m^2\left(d\left(\sum_{s\in T}\varphi^s\right), d\left(\sum_{t\in T}\varphi^t\right)\right)$$
$$= m^2 d^2 \sum_{s\in T}\left(\sum_{t\in T}(\varphi^s, \varphi^t)\right) = m^2 d^2 n,$$

since φ^s and φ^t are distinct irreducible characters of N whenever $s \neq t$. This proves (i).

It follows from Proposition 3.1 that $\mathrm{End}_{KN} V$ is K-algebra isomorphic to the given tensor product, so we need to show that A is a G/N-subalgebra of $\mathrm{End}_{KN} V$. Let $\rho : KG \to \mathrm{End}_K(V)$ be a representation affording V. Let $a \in A$, $f \in \mathrm{End}_{KG} V$, and $g \in G$ be a coset representative of $\bar{g} \in G/N$. Then

$$\begin{aligned}
f\left({}^{\bar{g}}a\right) &= f\left(\rho(g)a\rho(g)^{-1}\right) \\
&= \rho(g) f a \rho(g)^{-1} \quad (\text{since } f \in \mathrm{End}_{KG} V) \\
&= \rho(g) a f \rho(g)^{-1} \quad (\text{since } a \in C_{\mathrm{End}_{KN} V}(\mathrm{End}_{KG} V)) \\
&= \rho(g) a \rho(g)^{-1} f \quad (\text{since } f \in \mathrm{End}_{KG} V) \\
&= \left({}^{\bar{g}}a\right) f,
\end{aligned}$$

so ${}^{\bar{g}}a \in A$, for all $\bar{g} \in G/N$, and hence A is a G/N-subalgebra of $\mathrm{End}_{KN} V$. Since $\dim_K(\mathrm{End}_{KG} V) = m^2$ and $Z(\mathrm{End}_{KN} V) \cong K(\varphi)$, the other assertions now follow. □

The next theorem indicates that if the inertia group of an irreducible character of N is a normal subgroup of G, then the central simple G/N-algebras corresponding to quasihomogeneous KG-modules whose characters lie over a fixed irreducible character of N are all equivalent.

Theorem 3.2. *Let* N *be a normal subgroup of* G. *Let* $\varphi \in \mathrm{Irr}(N)$, *and let* $I = I_G(\varphi)$. *Suppose that* K *is any field containing* $\mathbb{Q}(\sum_t \varphi^t)$, *as* t *runs through a transversal of* I *in* G. *Then there is a unique equivalence class of central simple* G/N*-algebras which can be represented in the form* $[\mathrm{End}_{KN} V]$, *where* V *is a* χ*-quasiprimitive* KG*-module, for any* $\chi \in \mathrm{Irr}(G|\varphi)$.

Proof. If $\eta, \chi \in \mathrm{Irr}(G|\varphi)$ then η_N and χ_N are both integer multiples of $\sum_t \varphi^t$. Furthermore, $K(\eta_N) = K(\sum_t \varphi^t) = K(\chi_N)$. Thus η-quasihomogeneous KG-modules will also be χ-quasihomogeneous KG-modules, and vice-versa. The result then follows from Lemma 2.2. □

Using this theorem, whenever N is a normal subgroup of G and $\varphi \in \mathrm{Irr}(N)$, we can define a unique $[\![\varphi]\!] \in S(G/N, K)$, for any field K containing $\mathbb{Q}(\sum_t \varphi^t)$, by $[\![\varphi]\!] = [\![\chi]\!]$ for any $\chi \in \mathrm{Irr}(G|\varphi)$.

Now, suppose that G and H are finite groups with normal subgroups N and M such that $G/N \cong H/M$. Suppose further that $\varphi \in \mathrm{Irr}(N)$ and $\mu \in \mathrm{Irr}(M)$ with $I_G(\varphi) = I \trianglelefteq G$ and $I_H(\mu) = J \trianglelefteq H$, such that I/N is identified with J/M by the above isomorphism. Let T be a transversal of I in G, identified also with a transversal of J in H. Let K be any field for which $K(\sum_t \varphi^t) = K(\sum_t \mu^t) = K$. Then $[\![\varphi]\!], [\![\mu]\!] \in S(G/N, K)$. If one can show that $[\![\varphi]\!] = [\![\mu]\!]$, then it follows from Theorem 2.1 that there is a bijective character correspondence from $\mathrm{Irr}(G|\varphi) \to \mathrm{Irr}(H|\mu)$ that preserves Schur indices. Using Theorem 3.1, we can show that this occurs when there are characters of minimal degree (i.e., $d = 1$) that produce the same elements of the Brauer group of K.

Theorem 3.3. *Let G and H be finite groups, with normal subgroups N and M, respectively, satisfying $G/N \cong H/M$. Let $\varphi \in \mathrm{Irr}(N)$ and $\mu \in \mathrm{Irr}(M)$ with $K(\varphi) = K(\mu)$. Let $I = I_G(\varphi)$ and $J = I_H(\mu)$, and suppose that the isomorphism $G/N \cong H/M$ identifies I/N with J/M.*

Suppose φ extends to an $\alpha \in \mathrm{Irr}(I)$ for which α^G is quasiprimitive with respect to K, and that μ extends to a $\beta \in \mathrm{Irr}(M)$, for which β^H is quasiprimitive with respect to K. Suppose further that $[\alpha^G] = [\beta^H]$ in the Brauer group of K.

Then

(i) $[\![\varphi]\!] = [\![\mu]\!]$ *in* $S(G/N, K)$; *and*
(ii) *there is a bijective character correspondence* $\mathrm{Irr}(G|\varphi) \to \mathrm{Irr}(H|\mu)$ *that preserves Schur indices.*

Proof. Let V be a simple KG-module affording $m_K(\alpha^G)\alpha^G$ and U be a simple KH-module affording $m_K(\beta^H)\beta^H$.

If $[\alpha^G] = [\beta^H]$ in the Brauer group of K, then $\mathrm{End}_{KG} V \cong \mathrm{End}_{KH} U$ as central simple K-algebras. Since both have trivial $G/N(= H/M)$ action, they will thus be isomorphic as G/N-algebras. The fields $K(\varphi)$ and $K(\mu)$ are isomorphic, and the natural homomorphism from G/N into $\mathrm{Gal}(K(\varphi)/K)$ corresponds to the natural homomorphism from H/M into $\mathrm{Gal}(K(\mu)/K)$ since G/I can be identified with H/J. Therefore, these fields are isomorphic as G/N-algebras. By Theorem 3.2, $\mathrm{End}_{KN} V \cong \mathrm{End}_{KM} U$ as G/N-algebras over K. (i) then follows by the definitions of $[\![\varphi]\!]$ and $[\![\mu]\!]$, and (ii) holds by Theorem 2.1. □

Remark. Alexandre Turull has pointed out to the author that Theorem 3.3 is also an easy consequence of [7, Lemmas 3.6 and 3.7], by viewing $\mathrm{End}_{KN} V$ as a G/I-algebra, since G/I is abelian.

4. On Schur indices of quasiprimitive characters of solvable groups

We will now investigate the application of Theorem 3.3 to Schur index computation for solvable groups. We begin with some standard reductions. Let G be a solvable group and fix a faithful $\chi \in \mathrm{Irr}(G)$ for which we want to compute $m_{\mathbb{Q}}(\chi)$. Let p be a prime dividing $\chi(1)$, and let ζ be a primitive complex root of unity of order $\exp(G)$. Let $K \supseteq \mathbb{Q}(\chi)$ be the unique subfield of $\mathbb{Q}(\zeta)$ for which $[K : \mathbb{Q}(\chi)]$ is coprime to p and $[\mathbb{Q}(\zeta) : K]$ is a power of p. This implies that $m_K(\chi)$ will be equal to the largest power of p dividing $m_{\mathbb{Q}}(\chi)$.

Using Clifford theory, we can reduce to the situation where χ is faithful and quasiprimitive with respect to K, and so we may assume that every normal abelian subgroup of G is cyclic. By a well-known result of Philip Hall, such a group G has a nilpotent normal subgroup F of class at most 2 which is the central product of extraspecial q-groups for various primes q with a cyclic group Z. This subgroup F has index at most 2 in the Fitting subgroup of G, and $Z = Z(F)$ is a maximal abelian normal subgroup of G. In the case where $Z = F$, then the assumption that G is K-quasiprimitive implies that G is metabelian, and calculation of $m_K(\chi)$ is straightforward.

So, assume $Z \neq F$. If $\lambda \in \mathrm{Irr}(Z|\chi)$, then λ is a faithful linear character of Z, $I = I_G(\lambda) = C_G(Z)$ is a normal subgroup of G, and G/I is naturally isomorphic to $\mathrm{Gal}(K(\lambda)/K)$. Since F is nilpotent of class 2, λ is fully ramified with respect to F, so there is a unique $\varphi \in \mathrm{Irr}(F|\lambda)$ satisfying $\varphi_Z = e\lambda$ with $e = \sqrt{[F : Z]}$. Since φ vanishes off Z we have that $K(\varphi) = K(\lambda)$. By repeated use of the Isaacs–Dade character correspondence [2, Theorem B], we can show that there exists a subgroup J of I such that $I = FJ$, $F \cap J = Z$, and λ extends to J. As a consequence, φ extends to I, and these extensions can be used to define a bijective character correspondence between $\mathrm{Irr}(I|\varphi)$ and $\mathrm{Irr}(J|\lambda)$.

Theorem 4.1. *Let G be a solvable group and let $\chi \in \mathrm{Irr}(G)$. Suppose that χ is faithful, and that χ is quasiprimitive with respect to a field K containing $\mathbb{Q}(\chi)$ satisfying $[\mathbb{Q}(\zeta_{\exp(G)}) : K] = p^n$, for some fixed prime p.*

Let Z be a maximal abelian normal subgroup of G (hence cyclic). Let $\lambda \in \mathrm{Irr}(Z|\chi)$, and let $I = I_G(\lambda) = C_G(Z)$. Let $F = C_{F(G)}(Z)$, where $F(G)$ denotes the Fitting subgroup of G, and suppose $Z \neq F$. Let $\varphi \in \mathrm{Irr}(F)$ be the unique irreducible constituent of λ^F.

Let J be a subgroup of G chosen as above so that $I = FJ$, $F \cap J = Z$, and λ extends to J.

Then there exists a subgroup H of G such that $J \subseteq H$, $G = FH$, and $F \cap H = Z$. Furthermore, there is a bijective character correspondence between $\mathrm{Irr}(G|\varphi)$ and $\mathrm{Irr}(H|\lambda)$.

Proof. (The idea is to apply the construction of J given in [2, Section 7] to G, and use the fact that $I_G(\lambda)$ is normal in G to handle the problem of λ not being G-invariant.)

Let E be a normal subgroup of G chosen so that $Z \subseteq E$ and E/Z is a chief section of G. Let $C = C_I(E/Z)$. Note that $C = I \cap C_G(E/Z)$ so $C \trianglelefteq G$. The structure of G when it has a faithful quasiprimitive character implies that $C_{F/Z}(I/F) = 1$. Since F/Z is the direct sum of its G-chief factors, we have that $C_{E/Z}(I/C) = 1$. Since G is solvable, there exists a normal subgroup N of G such that $C \subseteq N \subseteq I$, $|N/C|$ is relatively prime to

$|E/Z|$, and $C_{E/Z}(N/C) = 1$. (This choice of N is also an allowable choice of N in Isaacs' construction.)

Let $B = C_I(F)$, so again $B \trianglelefteq G$. Since λ is I-invariant and fully ramified with respect to E, the same argument as in [2] shows that $BE = C$ and $B \cap E = Z$. Thus $C/B \cong E/Z$ is a normal Sylow subgroup of N/B. By the Schur–Zassenhaus theorem there is a subgroup R of G such that R/B is a complement of C/B in N/B, and every such complement of C/B is conjugate in N/B. We have that $E \cap R \subseteq E \cap R \cap C \subseteq E \cap B = Z$, and so $N = ER$. Since N is normal in G, we have that for all $g \in G$, $R^g \subset N$, and so R^g/B is a complement to C/B in N/B. Therefore, $R^g = R^x$, for some $x \in E$. It follows as in the Frattini argument that $G = EH$, where $H = N_G(R)$. Since $N_I(R)$ is also the group obtained in Isaac's construction, we can arrange that $J \subseteq H$. Finally, $[E \cap H, R] \subseteq E \cap R = Z$, and so $(E \cap H)/Z \subseteq C_{E/Z}(R/B) = 1$, which proves that $E \cap H = Z$.

If E_1/Z is another chief section of G contained in $(F \cap H)/Z$, then E_1/Z will be a chief section of H to which we can apply the same construction as above, using H in place of G. We repeat this process until the last relative complement H that is obtained satisfies $F \cap H = Z$.

Since $J \subseteq H$, the Isaacs–Dade character correspondence implies that λ extends to some $\beta \in \mathrm{Irr}(J)$, and φ extends to some $\alpha \in \mathrm{Irr}(I)$. The elements of $\mathrm{Irr}(I|\varphi)$ and of $\mathrm{Irr}(J|\lambda)$ are all those of the form $\alpha\delta$ and $\beta\delta$ as δ runs through $\mathrm{Irr}(I/F) = \mathrm{Irr}(J/Z)$, these characters being treated as characters of I and J, respectively. Every element of $\mathrm{Irr}(G|\varphi)$ is induced from I, hence is of the form $(\alpha\delta)^G$. Similarly, every element of $\mathrm{Irr}(H|\lambda)$ is induced from J. Thus the map taking $(\alpha\delta)^G$ to $(\beta\delta)^H$ is a bijection from $\mathrm{Irr}(G|\varphi)$ to $\mathrm{Irr}(H|\lambda)$. □

In order to be able to apply Theorem 3.3 to the groups G and H, we would need to be able to choose the extensions $\alpha \in \mathrm{Irr}(I|\varphi)$ and $\beta \in \mathrm{Irr}(J|\lambda)$ so that $[\alpha^G] = [\beta^H]$ in $\mathrm{Br}(K)$. In particular, this requires that $K(\alpha^G) = K(\beta^H) = K$. The next lemma gives a precise condition for this to occur.

Lemma 4.1. *Let G, F, φ, and I be as in Theorem* 4.1. *Let $\alpha \in \mathrm{Irr}(I)$ be an extension of φ. Then $K(\alpha^G) = K$ if and only if $K(\alpha) = K(\varphi)$ and for all $\sigma \in \mathrm{Gal}(K(\alpha)/K)$, $\alpha^\sigma = \alpha^t$, for some $t \in G$.*

Proof. Let T be a transversal of I in G. If $\sigma \in \mathrm{Gal}(K(\alpha)/K(\alpha^G))$, then $\alpha^G = (\alpha^G)^\sigma = (\alpha^\sigma)^G$, so α^σ is a constituent of $(\alpha^G)_I = \Sigma_{t \in T}\alpha^t$. Therefore, $\alpha^\sigma = \alpha^{t(\sigma)}$, for some $t(\sigma) \in T$. The map $\sigma \mapsto \overline{t(\sigma)}$ is a homomorphism from the abelian group $\mathrm{Gal}(K(\alpha)/K(\alpha^G))$ into the abelian group G/I, since $\alpha^{t(\sigma\tau)} = \alpha^{t(\tau\sigma)} = \alpha^{\tau\sigma} = (\alpha^\tau)^\sigma = (\alpha^{t(\tau)})^\sigma = (\alpha^\sigma)^{t(\tau)} = (\alpha^{t(\sigma)})^{t(\tau)} = \alpha^{t(\sigma)t(\tau)}$. Furthermore, this homomorphism is clearly injective.

Since α extends φ, we have that $K(\varphi) \subseteq K(\alpha)$. Thus in order for $K(\alpha^G)$ to be equal to K, we must have $K(\varphi) = K(\alpha)$, because otherwise $[K(\alpha) : K] > [K(\varphi) : K] = [G : I] \geqslant [K(\alpha) : K(\alpha^G)]$. When $K(\varphi) = K(\alpha)$, then $K(\alpha^G) = K$ exactly when the above homomorphism from $\mathrm{Gal}(K(\alpha)/K(\alpha^G)) = \mathrm{Gal}(K(\varphi)/K(\alpha^G))$ to $G/I \cong \mathrm{Gal}(K(\varphi)/K)$ is onto. This proves sufficiency in the statement of the lemma. Necessity is clear. □

The conditions of the lemma have been studied before. In [3], an extension satisfying both conditions of the lemma with respect to a field K and group G is referred to as a (K,G)-*standard extension*. An irreducible character α of a normal subgroup of a group G satisfying the second condition of the lemma with respect to the field K is called a K-*semi-invariant* character (with respect to G). In full generality, the lemma is equivalent to the statement that if $F \lhd G$, $\varphi \in \mathrm{Irr}(F)$ is K-semi-invariant in G, and α is an extension of φ to $I_G(\varphi)$, then $K(\alpha) = K$ if and only if α is a (K,G)-standard extension of φ. Thus the above lemma also applies in H to extensions of $\lambda \in \mathrm{Irr}(Z)$ to J.

One example of a (K,G)-standard extension is the case of a canonical extension. A *canonical* extension α of $\varphi \in \mathrm{Irr}(F)$ to I is one satisfying the condition $([I:F], o(\alpha)\alpha(1)) = 1$, where $o(\alpha)$ denotes the determinantal order of α. A canonical extension will always be a (K,G)-standard extension [3, Section 2]. In our situation, we will describe some conditions for the existence of a (K,H)-standard extension of λ to imply the existence of a (K,G)-standard extension of φ.

Since λ is linear, there is a precise condition for the existence of an extension β of λ satisfying $K(\beta) = K(\lambda)$, and for such a β to be a (K,H)-standard extension.

Proposition 4.1. *In the situation of Theorem* 4.1*, let* $L = K(\lambda)$.

(i) *There exists an extension* $\beta \in \mathrm{Irr}(J)$ *of* λ *satisfying* $L(\beta) = L$ *if and only if either* p *does not divide* $|Z|$ *or the order of the Sylow* p*-subgroup of any cyclic subgroup of* J/J' *containing* ZJ'/J' *does not exceed the number of* p*th power roots of unity in* L. (*Here* J' *denotes the commutator subgroup of* J.)

(ii) *If* β *is chosen as in* (i)*, then* β *is a* (K,H)*-standard extension if and only if* $\ker\beta \lhd H$.

Proof. Since λ is a faithful linear character of Z that extends to J, λ can be viewed as a faithful character of $ZJ'/J' \cong Z$ that extends to J/J'. If β is any extension of λ, then β may be viewed as a faithful irreducible character of a cyclic subgroup D of J/J' containing ZJ'/J'. Suppose p^k is the number of pth power roots of unity in L. Since p divides $\exp(G)$, our choice of K implies that there is a pth root of unity in K, so $k \geqslant 1$. If $L \neq L(\beta)$, then $[L(\beta):L]$ is a non-trivial power of p. Since $L(\beta)$ and L are both cyclotomic extensions of K, this means either that $p^{(k+1)}$ divides $|D|$, or that there exists a prime q dividing $|D|$ such that $p^{(k+1)}$ divides $(q-1)$. In the latter case, the indicated prime q would not divide $|Z|$, and so $D = R \times S$, where S is the cyclic Sylow q-subgroup of D and $ZJ'/J' \subseteq R$. If λ extends to $\mu \in \mathrm{Irr}(R)$ with $L(\mu) = L$, then $\mu \times 1_S$ is an extension of λ to D whose field of character values is L. Thus the latter situation does not prevent us from being able to choose an appropriate extension of λ. If $p^{(k+1)}$ divides $|D|$, but $|Z|$ is not divisible by p, we can again split $D = R \times S$, where S is a cyclic Sylow p-subgroup of D and $ZJ'/J' \subseteq R$. As in the previous case, an appropriate extension to D will exist provided there is an appropriate extension to R. If $p^{(k+1)}$ divides $|D|$, and p divides $|Z|$, then every extension of λ to D will be faithful on the Sylow p-subgroup of D. Thus there is no appropriate choice of β exactly when this situation occurs. This proves (i).

Let $\beta \in \mathrm{Irr}(J)$ be an extension of λ with $K(\beta) = K(\lambda)$, and let $B = \ker\beta$. Let T be a transversal of J in H. If B is not normal in H, then there is a $t \in T$ such that $B \neq B^t = \ker(\beta^t)$. The corresponding $\bar{t} \in H/J$ cannot be in the image of the map from

$\mathrm{Gal}(K(\beta)/K(\beta^H))$ defined in Lemma 4.1. Since this map is not onto, β cannot be a (K,H)-standard extension of λ. On the other hand, if B is normal in H, then J/B is a cyclic normal subgroup of H/B. If $J/B = \langle x\rangle$, then for all $\bar{t} \in H/J$, $x^t = x^{i(t)}$, for some integer $i(t)$ relatively prime to $o(x)$. Thus $\beta^t(x) = \beta(x)^{i(t)^{-1}}$. Since automorphisms of $K(\beta)$ are determined by their action on $\beta(x)$, this shows that $\beta^t = \beta^\sigma$, for some $\sigma \in \mathrm{Gal}(K(\beta)/K(\beta^H))$, and so the map $\mathrm{Gal}(K(\beta)/K(\beta^H))$ into H/J is actually onto. (ii) then follows from the assumption that $K(\beta) = K(\lambda)$ and Lemma 4.1. $\square$

The next proposition shows that, as long as $m_K(\varphi) = 1$, the existence of an extension of λ having a smallest possible field of character values over K implies that φ also has such an extension.

Proposition 4.2. *In the situation of Theorem* 4.1*, let $L = K(\lambda)$ and assume $m_L(\varphi) = 1$. Suppose $\lambda \in \mathrm{Irr}(Z|\chi)$ has an extension $\beta \in \mathrm{Irr}(J)$ with $L(\beta) = L$. Then $\varphi \in \mathrm{Irr}(F|\lambda)$ will also have an extension $\alpha \in \mathrm{Irr}(I)$ with $L(\alpha) = L$.*

Proof. Since φ is I-invariant, we can apply the methods of stable Clifford theory (for notation see [5, Section 1]) to establish a condition for the existence of α.

Let $S = I/F = J/Z$, and let $\{t_x\}_{x\in S}$ be a transversal of Z in J (hence of F in I) with $t_1 = 1$. For $x, y \in S$, define $t(x,y) = t_x t_y t_{xy}^{-1}$, so that t is the factor set with values in Z induced by this transversal.

Let V be an irreducible LF-module affording φ. Stable Clifford theory implies that

$$\mathrm{End}_{LI}\big(V^I\big) = \bigoplus_{x\in S} E_x,$$

where the elements of E_x map $V = V\otimes 1$ to $V_x = V \otimes t_x$, for all $x \in S$. Thus $E_1 \cong \mathrm{End}_{LF}(V) \cong L$. Since E_1 is a central simple L-algebra, there always exists units T_x lying in E_x which centralize E_1 and which satisfy $E_x = E_1 T_x$, for all $x \in S$. In our case, E_1 is a field, so we may choose the units T_x to be the maps $v \mapsto v \otimes t_x$, for all $v \in V = V \otimes 1$. The Clifford extension

$$\Omega_{LF}(\varphi) = \bigcup_{x\in S} \big(L^\times\big) T_x$$

is an extension of the form

$$1 \to L^\times \to \Omega_{LF}(\varphi) \to S \to 1$$

with factor set given by $T(x,y) = T_x T_y T_{xy}{}^{-1}$, for all $x, y \in S$. (Here we view $T(x,y)$ as being the scalar associated with the map $T_x T_y T_{xy}{}^{-1} \in E_1$, for all $x, y \in S$.) For $v \in V$ $(= V \otimes 1)$, we have for all $x, y \in S$ that $T_x T_y T_{xy}{}^{-1}(v) = \lambda(t(x,y))1_V(v)$, since the action of any $z \in Z = Z(F)$ on the e-dimensional module V is scalar multiplication by a multiple of the identity map and $\varphi_Z = e\lambda$ implies that the scalar is $\lambda(z)$. Thus $T(x,y) = \lambda(t(x,y))$, for all $x, y \in S$. The relevance of the Clifford extension to our situation is that the cohomology class $\omega_{LF}(\varphi) \in H^2(S, L^\times)$ of T is trivial if and only if there exists $\alpha \in \mathrm{Irr}(I)$ such that $\alpha_F = \varphi$ and $L(\alpha) = L$ [5, (1.1)].

Since $\beta \in \mathrm{Irr}(J)$ is an extension of λ with $L(\beta) = L$, we know that the corresponding $\omega_{LZ}(\lambda)$ must be trivial. This cohomology class comes from the Clifford extension defined in terms of an irreducible LZ-module U. Since λ is J-invariant, we have

$$\mathrm{End}_{LJ}\bigl(U^J\bigr) = \bigoplus E(U)_x,$$

where the elements of $E(U)_x$ map $U = U \otimes 1$ to $U_x = U \otimes t_x$, and $E(U)_1 \cong \mathrm{End}_{LZ}(U) \cong L$. Since λ is linear, the factor set given by $f(x, y) = \lambda(t(x, y))$ for all $x, y \in S$ is a representative of the cohomology class $\omega_{LZ}(\lambda)$ [5, (1.5)]. Since the factor set f is clearly equal to T, we have that $\omega_{LF}(\varphi) = \omega_{LZ}(\lambda) = 1$, which implies the result. $\square$

Since F is nilpotent, the only situation in which $m_L(\varphi) > 1$ can occur is when $[F : Z]$ is even, and L does not contain a fourth root of unity. In this case the following construction takes care of the difficulty. Replace the group G by $\widehat{G} =$ the central product of G with D_4 (the dihedral group of order 8). Then Z gets replaced by the maximal cyclic subgroup ZC_4, where C_4 is the cyclic normal subgroup of order 4 in D_4. There exists $\hat{\lambda} \in \mathrm{Irr}(ZC_4)$ such that $L(\hat{\lambda}) = L(\zeta_4)$, $\hat{\lambda}$ is fully ramified with respect to FC_4, and $I_{\widehat{G}}(\hat{\lambda}) = IC_4$. Thus there exists a unique $\hat{\varphi} \in \mathrm{Irr}(FC_4|\hat{\lambda})$, and $m_L(\hat{\varphi}) = 1$ since $\zeta_4 \in L(\hat{\varphi})$. If λ extends to $\beta \in \mathrm{Irr}(J)$ with $L(\beta) = L$, then there will be an extension $\hat{\beta} \in \mathrm{Irr}(JC_4)$ of $\hat{\lambda}$ and β satisfying $L(\hat{\beta}) = L(\hat{\lambda})$. Thus we can conclude from the above that $\hat{\varphi}$ extends to $\hat{\alpha} \in \mathrm{Irr}(IC_4)$ with $L(\hat{\alpha}) = L(\hat{\varphi})$. The Schur index situation still has not changed because each $\chi \in \mathrm{Irr}(G|\varphi)$ has a unique faithful $\hat{\chi} \in \mathrm{Irr}(\widehat{G}|\chi)$ with $\hat{\chi}(1) = 2\chi(1)$ and $m_K(\hat{\chi}) = m_K(\chi)$. Furthermore, if β is a (K, H)-standard extension of λ, then $\hat{\beta}$ will be $(K, \widehat{H})$-standard extension of $\hat{\lambda}$ to JC_4. This is because $H/J \cong \mathrm{Gal}(K(\beta)/K)$ will act trivially on an element of order 4 in C_4, and thus not interfere with the action of D_4/C_4 on C_4. For the same reason, if $\alpha \in \mathrm{Irr}(I)$ is a (K, G)-standard extension of $\varphi \in \mathrm{Irr}(F)$, then $\hat{\alpha} \in \mathrm{Irr}(IC_4)$ will be a $(K, \widehat{G})$-standard extension of $\hat{\varphi} \in \mathrm{Irr}(FC_4)$.

When extensions α of φ and β of λ exist, they define a character ψ of J with $Z \subseteq \ker\psi$ by letting $\alpha_J = \beta\psi$. This ψ is known as the *Isaacs character of J/Z arising from α and β*.

Lemma 4.2. *In the situation of Theorem* 4.1, *suppose that $m_K(\varphi) = 1$ and that λ has a (K, H)-standard extension $\beta \in \mathrm{Irr}(J)$. Let the natural isomorphism from $\mathrm{Gal}(K(\beta)/K)$ to H/J be denoted by $\sigma \mapsto \overline{t(\sigma)}$. Let $\alpha \in \mathrm{Irr}(I)$ be an extension of φ such that $K(\alpha) = K(\varphi)$* (*which exists by Proposition* 4.2). *Suppose that the Isaacs character ψ of J/Z arising from α and β has the property that for all $\sigma \in \mathrm{Gal}(K(\beta)/K)$, $\psi^\sigma = \psi^{t(\sigma)}$. Then α is a (K, G)-standard extension of φ.*

Proof. Let T be a transversal of J in H, hence also of I in G. Use T to identify G/I with H/J. We need to show that for all $\sigma \in \mathrm{Gal}(K(\alpha)/K)$, $\alpha^\sigma = \alpha^t$ for some $\bar{t} \in G/I$. We have $(\alpha^\sigma)_J = (\alpha_J)^\sigma = (\beta\psi)^\sigma = \beta^\sigma\psi^\sigma$, and since J is normal in H, $(\alpha^{t(\sigma)})_J = (\alpha_J)^{t(\sigma)} = (\beta\psi)^{t(\sigma)} = \beta^{t(\sigma)}\psi^{t(\sigma)}$. Since $\psi^\sigma = \psi^{t(\sigma)}$ is the Isaacs character of J/Z arising from the extensions α^σ of φ^σ and β^σ of λ^σ, it defines a bijection between $\mathrm{Irr}(I|\varphi^\sigma)$ and $\mathrm{Irr}(J|\lambda^\sigma)$. Thus $(\alpha^\sigma)_J = (\alpha^{t(\sigma)})_J$ implies $\alpha^\sigma = \alpha^{t(\sigma)}$, as required. $\square$

One situation where the condition of Lemma 4.2 is guaranteed to hold is the case where the Isaacs character ψ arising from an appropriate choice of α and β happens to be canonical as defined in [4, Definition 5.2]. This is because determinantal order and multiplicity of trivial characters on restriction to Sylow subgroups will not be affected by Galois conjugacy or conjugation in H. Every ψ^{σ} and ψ^{t} will be equal to ψ only when ψ is K-valued and constant on the H-conjugacy classes of J.

When standard extensions α of φ and β of λ exist, then we do have that $[\alpha^G], [\beta^H] \in \mathrm{Br}(K)$. One way to ensure that $[\alpha^G] = [\beta^H]$ is to have the multiplicity $((\alpha^G)_H, \beta^H)$ relatively prime to p. If ψ is the Isaacs character of J/Z arising from α and β, then

$$\left(\left(\alpha^G\right)_H, \beta^H\right) = \left((\alpha_J)^H, \beta^H\right) = (\alpha_J, \beta) = (\beta\psi, \beta) = (\psi, 1_J).$$

Thus we only need to check that $(\psi, 1_J)$ is relatively prime to p. (Examples satisfying $(\psi, 1_J) = 1$ and $(\psi, 1_J) = 0$ are described in [1].) We can now summarize the application of Theorem 3.3 to solvable groups that we have just described.

Theorem 4.2. *Let G, χ, K, Z, λ, F, φ, I, and J be as in the assumptions of Theorem* 4.1. *Let $H \subset G$ be chosen as in Theorem* 4.1 *so that $G = FH$, $F \cap H = Z$, and λ extends to $J = I \cap H$. Suppose that λ has a (K, H)-standard extension $\beta \in \mathrm{Irr}(J)$. Let $\sigma \mapsto \overline{t(\sigma)}$ be a fixed isomorphism from* $\mathrm{Gal}(K(\beta)/K)$ *onto $H/J = G/I$. Suppose that $m_K(\varphi) = 1$, and let $\alpha \in \mathrm{Irr}(I|\varphi)$ be such that $\alpha_F = \varphi$ and $K(\alpha) = K(\varphi)$. Let ψ be the character of J/Z defined by $\alpha_J = \beta\psi$. Suppose that for all $\sigma \in \mathrm{Gal}(K(\beta)/K)$, $\psi^{\sigma} = \psi^{t(\sigma)}$, and that $(\psi, 1_J)$ is relatively prime to p.*

Then

- (i) $[\alpha^G] = [\beta^H]$ *in the Brauer group of K;*
- (ii) $[\![\varphi]\!] = [\![\lambda]\!]$ *in $S(G/F, K) = S(H/Z, K)$; and*
- (iii) *there is a bijective character correspondence* $\mathrm{Irr}(G|\varphi) \to \mathrm{Irr}(H|\lambda)$ *that preserves Schur indices and elements of corresponding Brauer groups. In particular, there exists a $\xi \in \mathrm{Irr}(H)$ such that* $[\xi] = [\chi] \in \mathrm{Br}(K)$.

Proof. The results of this section show that these assumptions imply that $K(\alpha^G) = K(\beta^H) = K$ and that $((\alpha^G)_H, \beta^H)$ is relatively prime to p. Since our choice of K implies that $\mathrm{Br}(K)$ is a p-group, it follows from the Brauer–Witt theorem that $[\alpha^G] = [\beta^H]$ in $\mathrm{Br}(K)$. The other two conclusions then follow from Theorem 3.3. □

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